Fibration structures and formulae for the Euler characteristics of Milnor fibers (Theory of singularities of smooth mappings and around it)

Dutertre, Nicolas; Araújo dos Santos, Raimundo; Chen, Ying; Andrade do Espirito Santo, Antonio

RIMS Kôkyûroku Bessatsu
B55 (2016), 41–65

Fibration structures and formulae for the Euler characteristics of Milnor fibers

By

Nicolas Dutertre*, Raimundo Araújo dos Santos**, Ying Chen***
and Antonio Andrade do Espirito Santo†

Abstract

In this article, we review some fibration structure theorems (also called Milnor's fibration theorems) recently proved in the real and complex cases, in the local and global settings. We give several Poincaré-Hopf type formulae which relate the Euler-Poincaré characteristics of these fibers (also called Milnor fibers) and indices (topological degrees) of appropriate vector fields defined on spheres of radii small or big enough. Some of them are new formulas.

§ 1. Introduction

In this paper, we provide a brief review of some extensions of the real and complex Milnor fibration theorems for isolated and non-isolated singularities and we give some

Received September 1, 2014. Revised October 16, 2015.
2010 Mathematics Subject Classification(s): 32S55, 14P15, 57R45, 58K05
Key Words: *Milnor fibrations, Open book structures, Euler characteristic, Semi-algebraic sets.*
*Aix-Marseille Université, CNRS, Centrale Marseille, UMR 7373, 13453 Marseille, France.
e-mail: nicolas.dutertre@univ-amu.fr
**R. Araújo dos Santos was partially supported by Fapesp project 2013/23443-5 and CNPq project 474701/2012-3. Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos, SP, Brazil.
e-mail: rnonato@icmc.usp.br
***Y. Chen thanks Brazilian Scholarships support by Fapesp project 2012/18957-7. Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos, SP, Brazil.
e-mail: ychenmaths@hust.edu.cn
†Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos, SP, Brazil, and, Centro de Ciências Exatas e Tecnológicas, Universidade Federal do Recôncavo da Bahia - Campus de Cruz das Almas, Rua Rui Barbosa, 710, Centro, CEP 44.380-000 - Cruz das Almas, Ba, Brazil.
e-mail: andrade@ufrb.edu.br

old or new Poincaré-Hopf type formulae which give a connection between the Euler-Poincaré characteristic of the (regular) Milnor fiber and the index of a gradient vector field.

The paper is organized as follows. In Section 2, we first remind some fibration structure theorems which extend the very well known complex and real local Milnor's fibrations theorems stated in [24]. In the sequel, we remind Milnor's formula for complex isolated singularities (Poincaré-Hopf type formula) and we state several extensions of such formula which have been developed recently for functions and mappings with non-isolated singularities, in the complex and real cases.

In Section 3, we try to follow the line of development of Section 2, i.e., we also start introducing and showing some fibration theorems, but now in the global case (at infinity), and in the sequel we show several new Poincaré-Hopf type formulae in this setting.

The authors would like to acknowledge Professor Takashi Nishimura for inviting them to submit this paper to RIMS proceedings. Thanks ! They also thank the referee for his careful reading.

§ 2. Fibration structure and degree formulae - Local setting

§ 2.1. Milnor's fibration structure

For a germ of complex holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, in [24, Theorem 4.8, page 43] Milnor proved that there exists a small enough $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$ the projection

$$\frac{f}{|f|} : S_\epsilon^{2n+1} \setminus \{f = 0\} \to S^1 \tag{2.1}$$

is a smooth locally trivial fibration, where S_ϵ^{2n+1} is the sphere of radius ϵ centered at the origin in $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$.

Later, it was proved by Lê D. Tráng [21] that the restriction map

$$f_| : \mathring{B}_\epsilon^{2n+2} \cap f^{-1}(S_\delta^1) \to S_\delta^1$$

is also the projection of a locally trivial smooth fibration and that such a fibration is fiber-equivalent to the fibration (2.1), where $\mathring{B}_\epsilon^{2n+2}$ denotes the interior of the closed ball of radius ϵ centered at the origin B_ϵ^{2n+2} and S_δ^1 is the circle of radius δ in $\mathbb{C}$.

For a real analytic map germ $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n > p \geq 2$ with an isolated singularity at the origin, Milnor also proved in [24, Theorem 11.2, page 97] that, for all $\epsilon > 0$ small enough, there exists $0 < \eta \ll \epsilon$ such that the restriction map

$$F_| : B_\epsilon^n \cap F^{-1}(S_\eta^{p-1}) \to S_\eta^{p-1}$$

is the projection of a locally trivial smooth fibration (called fibration in the "Milnor tube"), with fiber $\mathcal{M}_F^T := (F_|)^{-1}(y)$, $y \in S_\eta^{p-1}$. He also constructed a convenient vector field in $B_\epsilon^n \setminus V$, where $V = F^{-1}(0)$, and used its flows to push the Milnor tube $B_\epsilon^n \cap F^{-1}(S_\eta^{p-1})$ to the sphere S_ϵ^{n-1}, but keeping the boundary $S_\epsilon^{n-1} \cap F^{-1}(S_\eta^{p-1})$ fixed. This way he got a new locally trivial smooth fibration from $S_\epsilon^{n-1} \setminus F^{-1}(\overset{\circ}{B}{}_\eta^p)$ to S^{p-1}, after rescaling the radius of the sphere in the target space. Now, since 0 is an isolated critical point of F, it is not hard to see that it is possible to extend this new fibration to get a smooth locally trivial fibration

$$S_\epsilon^{n-1} \setminus \mathrm{Lk}^0(V) \to S^{p-1},$$

where $\mathrm{Lk}^0(V) := V \cap S_\epsilon^{n-1}$ is called the link of the zero set V at the origin.

As Milnor pointed out, in general we cannot expect that the projection of this last fibration will be given by the canonical one $\frac{F}{\|F\|}$. In [5] the authors provide a characterization of such a fibration in the more general case of non-isolated singularities (see also [28, 1, 9]) as follows.

Theorem 2.1.
Let $F = (f_1, \ldots, f_p) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n > p \geq 2$, be a germ of an analytic mapping and Σ_F its singular locus i.e., the set of points where the gradients $\nabla f_1, \ldots, \nabla f_p$ are linearly dependent. Assume that $\Sigma_F \cap V \subseteq \{0\}$. The following two statements are equivalent:

1. *for all $\epsilon > 0$ small enough the projection $\frac{F}{\|F\|} : S_\epsilon^{n-1} \setminus \mathrm{Lk}^0(V) \to S^{p-1}$ is a smooth locally trivial fibration.*
2. *for all $\epsilon > 0$ small enough the projection $\frac{F}{\|F\|} : S_\epsilon^{n-1} \setminus \mathrm{Lk}^0(V) \to S^{p-1}$ is a smooth submersion.*

We remark that, this equivalence is not so trivial because if the link is not empty such a projection mapping is never proper.

The Milnor tube fibration for non-isolated singularity was established by D. Massey in [23] as follows.

Let $F = (f_1, \ldots, f_p) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a smooth map germ, $2 \leq p \leq n-1$, and ρ be the square of the distance function to the origin and denote by $M(F)$ the set of critical points of the pair (F, ρ), i.e., the set of points where the gradients $\nabla\rho, \nabla f_1, \ldots, \nabla f_p$ are linearly dependent. This set is called the *Milnor set* and it also plays an important role in the study of asymptotical behavior of polynomial mappings at infinity, as the reader will see in the next section.

It follows by definition that $\Sigma_F \subseteq M(F)$.

Definition 2.2. [23] Let F and ρ be as above.

1. We say that F satisfies *Milnor's condition (a)* at the origin, if $\Sigma_F \subset V$ in a neighborhood of the origin.

2. We say that F satisfies *Milnor's condition (b)* at the origin, if $V \cap \overline{M(F) \setminus V} \subset \{0\}$ in a neighborhood of the origin, where the notation $\overline{X}$ means the topological closure of the space X.

Remark. It follows from Definition 2.2 the equivalence:
The mapping F satisfies Milnor's condition (b) at the origin if and only if for each $\epsilon > 0$ small enough, there exists $\delta > 0$, $0 < \delta \ll \epsilon$ such that the restriction map $F_| : S_\epsilon^{n-1} \cap F^{-1}(B_\delta^p \setminus \{0\}) \to B_\delta^p \setminus \{0\}$ is a smooth submersion (and onto, if the link of $F^{-1}(0)$ is not empty).

We say that $\epsilon > 0$ is a *Milnor radius for F at the origin*, provided that $B_\epsilon^n \cap (\overline{\Sigma_F - V}) = \varnothing$, and $B_\epsilon^n \cap V \cap (\overline{M(F) \setminus V}) \subseteq \{0\}$, where B_ϵ^n denotes the closed ball in $\mathbb{R}^n$ with radius ϵ.

Consequently, under Milnor's conditions (a) and (b), we can conclude that for all regular values close to the origin the respective fibers in the closed ϵ-ball are smooth and transverse to the sphere S_ϵ^{n-1}.

Theorem 2.3 ([23], Theorem 4.3, page 284). *Let $F : (f_1, \dots, f_p) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a smooth map germ satisfying Milnor's conditions (a) and (b) and $\epsilon_0 > 0$ be a Milnor radius for F at the origin. Then, for each $0 < \epsilon \leq \epsilon_0$, there exists δ, $0 < \delta \ll \epsilon$, such that*

$$F_| : B_\epsilon^n \cap F^{-1}(B_\delta^p \setminus \{0\}) \to B_\delta^p \setminus \{0\} \tag{2.2}$$

is the projection of a smooth locally trivial fiber bundle.

Proof. (Idea) Since $\epsilon_0 > 0$ is a Milnor radius for F at the origin, we have that $\Sigma_F \cap B_{\epsilon_0}^n \subset V \cap B_{\epsilon_0}^n$. It means that, for all $0 < \epsilon \leq \epsilon_0$ the map $F_| : \mathring{B}_\epsilon^n \setminus V \to \mathbb{R}^p$ is a smooth submersion in the open ball $\mathring{B}_\epsilon^n$.

From the Milnor condition (b), and the remark above, it follows that: for each ϵ there exists δ, $0 < \delta \ll \epsilon$, such that

$$F_| : S_\epsilon^{n-1} \cap F^{-1}(B_\delta^p - \{0\}) \to B_\delta^p - \{0\}$$

is a submersion on the boundary S_ϵ^{n-1} of the closed ball B_ϵ^n.

Now, combining these two conditions we have that, for each ϵ, we can choose δ such that

$$F_| : B_\epsilon^n \cap F^{-1}(B_\delta^p - \{0\}) \to B_\delta^p - \{0\}$$

is a proper smooth submersion. Applying the Ehresmann Fibration Theorem for the manifold with boundary $B^n_\epsilon \cap F^{-1}(B^p_\delta - \{0\})$, we get that it is a smooth locally trivial fibration. $\square$

Corollary 2.4 (Milnor's fibration in the tube)**.** *Let $F = (f_1, \dots, f_p) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a smooth map germ satisfying Milnor's conditions (a) and (b) and $\epsilon_0 > 0$ be a Milnor radius for F at the origin. Then, for each $0 < \epsilon \leq \epsilon_0$, there exists δ, $0 < \delta \ll \epsilon$, such that*

$$F_| : B^n_\epsilon \cap F^{-1}(S^{p-1}_\delta) \to S^{p-1}_\delta \tag{2.3}$$

is the projection of a smooth locally trivial fiber bundle.

Example 2.5. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ, then it satisfies the Milnor conditions (a) and (b).

In fact, Milnor's condition (a) and (b) can be seen as an application of Łojasiewicz's inequality [22] which states that, in a small neighborhood of the origin, there are constants $C > 0$ and $0 < \theta < 1$ such that

$$|f(x)|^\theta \leq C\|\nabla f(x)\|.$$

So, Milnor's condition (a) follows. In [19], page 323, Hamm and Lê proved that the Łojasiewicz inequality implies the Thom a_f-condition for a Whitney (a) stratification of V. Therefore, Milnor's condition (b) follows. A proof of these facts is given in a more general setting in [23], Lemma 5.4 and Theorem 5.5.

Example 2.6. Let $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be an analytic map-germ with an isolated singular point at the origin. By an application of the Curve Selection Lemma, Milnor's condition (a) holds (see for example [13], Lemma 3.1). Milnor's condition (b) also holds since the zero locus of F is transversal to all small spheres. This latter fact can be also proved using [13], Lemma 3.1.

Example 2.7. The next example comes from [2], Example 5.1, Section 5. Let $f = (P, Q) : (\mathbb{R}^5, 0) \to (\mathbb{R}^2, 0)$, $f(x, y, z, u, v) = (y^4 - z^2x^2 - x^4 + u^2 - v^2, xy + 2uv)$. It is easy to see that $\Sigma_f \subseteq V$. It was proved in [2] that Milnor's condition (b) follows as an application of the Curve Selection Lemma.

In [5, 2] the authors considered the following definition.

Definition 2.8 ([2])**.** We say that the pair (K, θ) is a *higher open book structure with singular binding* on an analytic manifold M of dimension $m - 1 \geq p \geq 2$ if $K \subset M$ is a singular real analytic subvariety of codimension p and $\theta : M \setminus K \to S^{p-1}_1$ is a locally trivial smooth fibration such that K admits a neighbourhood N for which the restriction $\theta_{|N \setminus K}$ is the composition $N \setminus K \xrightarrow{h} B^p \setminus \{0\} \xrightarrow{\pi} S_1^{p-1}$ where h is a locally trivial fibration and $\pi(s) = s/\|s\|$.

In such a case, one says that the *singular subvariety* K is the *binding* and that the (closures of) the fibers of θ are the *pages* of the *open book*.

Let us denote by $M(\frac{F}{\|F\|})$ the singular locus of the pair $(\frac{F}{\|F\|}, \rho)$, where $\rho(x) = x_1^2 + \cdots + x_n^2$ is the square of distance function to the origin. Inspired by Milnor's conditions in [23], in the paper [2] the authors proved some theorems about fibration structures on spheres of radii small enough for non-isolated singularities, as follows.

Theorem 2.9 ([2], Theorem 1.3, page 819). *Let $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a real analytic mapping germ. Assume that Milnor's condition (b) holds and codim$_{\mathbb{R}} V = p$. If the set $M(\frac{F}{\|F\|})$ is empty, then for all $\epsilon > 0$ small enough, the pair $(\mathrm{Lk}^0(V), \frac{F}{\|F\|})$ is a higher open book structure on the spheres S_ϵ^{n-1}, with singular binding. In particular, the projection*

$$\frac{F}{\|F\|} : S_\epsilon^{n-1} \setminus \mathrm{Lk}^0(V) \to S^{p-1} \tag{2.4}$$

is a smooth locally trivial fibration.

It seems natural to ask the following question: how does the fibers of fibrations (2.3) and (2.4) relate with each other, once both fibrations exist ?

Denote by $\mathcal{M}_F^T$ and $\mathcal{M}^S_{\frac{F}{\|F\|}}$ the fibers of fibrations (2.3) and (2.4), respectively. In [9] the authors considered a condition called d-regularity and assuming also that F satisfies a Thom a_F-condition, they showed that the both fibrations are fiber-equivalent. We should point out that, it is pretty clear that the Thom a_F-condition along V implies Milnor's condition (b) (see [23], Lemma 5.4 and Theorem 5.5). But, the converse is not true in general as the reader can check in [2], section 5.3, page 827, in an example provided by A. Parusiński.

More recently, just assuming Milnor's condition (b) and using a quite different machinery, the authors in [14] proved a result similar to the one stated in [9]. This result was proved first by M. Oka in [27] for non-degenerate and convenient mixed polynomial germs.

Proposition 2.10 ([14], section 5.0.1, Proposition 5.2). *Let*

$$F = (f_1, \ldots, f_p) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$$

be a real analytic map germ satisfying Milnor's conditions (a) and (b). If $M(\frac{F}{\|F\|})$ is empty (as a germ of set), then the two fibers $\mathcal{M}_F^T$ and $\mathcal{M}^S_{\frac{F}{\|F\|}}$ are homotopy equivalent.

At this point, we would like to invite the interested reader to look at section 5.0.1 of [14] for further details.

§ 2.2. A degree formula in the complex case

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function. In the case that $0 \in \mathbb{C}^{n+1}$ is an isolated critical point of f, Milnor considered the topological degree, denoted by $\deg_0 \nabla f$, of the normalized gradient vector field

$$\epsilon \frac{\nabla f}{\|\nabla f\|} : S_\epsilon^{2n+1} \to S_\epsilon^{2n+1}, \tag{2.5}$$

where $\nabla f = (\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_{n+1}})$. Let us denote by $\mu_f := \deg_0 \nabla f$.

This number was named later as "Milnor number" and it has several important meanings. For instance, it is known that the regular fiber $F_\theta = (\frac{f}{\|f\|})^{-1}(\theta)$, $\theta \in S^1$, has the homotopy type of a bouquet of n-dimensional spheres and the number of spheres in the bouquet is μ_f. Therefore, the Euler-Poincaré characteristic of the (regular) fiber satisfies the following *Poincaré-Hopf formula*

$$\chi(F_\theta) = 1 + (-1)^n \mu_f. \tag{2.6}$$

In the search of topological and analytical invariants of complex and real singularities, this type of Poincaré-Hopf formula became a starting point of several other formulae, for isolated and non-isolated singularities, in the local and global settings. Below we will discuss briefly some of them in the local real setting.

§ 2.3. Degree formulae in the real case

In [20] Khimshiashvili considered a germ of a real analytic function $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with an isolated critical point at the origin, and proved that

$$\chi\big(f^{-1}(\delta) \cap B_\epsilon^n\big) = 1 - \mathrm{sign}(-\delta)^n \deg_0 \nabla f, \tag{2.7}$$

where δ with $0 < |\delta| \ll \varepsilon \ll 1$ is a regular value, B_ϵ^n stands for the closed ball with radius ϵ centered at the origin, ∇f is the gradient vector field of f and $\deg_0 \nabla f$ is the topological degree of the mapping

$$\epsilon \frac{\nabla f}{\|\nabla f\|} : S_\epsilon^{n-1} \to S_\epsilon^{n-1}.$$

The formula above was extended for mappings with isolated singularities as follows.

Let $F = (f_1, \cdots, f_p) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be an analytic mapping with an isolated critical point at the origin. Then, each coordinate function f_i, for $i = 1, \ldots, p$, also has an isolated critical point at the origin. Using these notations, formulae (2.6) and (2.7) above were extended in the following way:

Theorem 2.11 ([4], Proposition 3.3, page. 71). *Let $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n > p \geq 1$, be as above. The following holds true:*

(i) If n is even, then $\chi(\mathcal{M}_F^T) = 1 - \deg_0 \nabla f_1$. Moreover, we have

$$\deg_0 \nabla f_1 = \deg_0 \nabla f_2 = \cdots = \deg_0 \nabla f_p.$$

(ii) If n is odd, then $\chi(\mathcal{M}_F^T) = 1$. Moreover, we have $\deg_0 \nabla f_i = 0$ for $i = 1, 2, \ldots, p$.

We remark that in [18] Hamm proved that the formulae above still holds true if one changes the germ of space $(\mathbb{R}^n, 0)$ by $(X, 0)$ where X is a pure n-dimensional real analytic space with isolated singularity at the origin. In [14] the authors proved several formulae which in turn extend the results of [18].

Example 2.12 (Case of isolated singularities of holomorphic functions). Given a germ of a holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated critical point at the origin, we consider the complex variable $z_j = x_j + iy_j$, for $0 \leq j \leq n$ and $f = (P, Q) : (\mathbb{R}^{2n+2}, 0) \to (\mathbb{R}^2, 0)$ where $P = \Re(f)$ and $Q = \Im(f)$ are the real and imaginary parts of f, respectively. By Cauchy-Riemann equations, we have that

$$\nabla P(x,y) = \left(\frac{\partial P(x,y)}{\partial x_0} - i\frac{\partial P(x,y)}{\partial y_0}, \ldots, \frac{\partial P(x,y)}{\partial x_n} - i\frac{\partial P(x,y)}{\partial y_n} \right),$$

which we can be identified with

$$\nabla P(x,y) = \left(\frac{\partial P(x,y)}{\partial x_0}, -\frac{\partial P(x,y)}{\partial y_0}, \ldots, \frac{\partial P(x,y)}{\partial x_n}, -\frac{\partial P(x,y)}{\partial y_n} \right).$$

Let us write $H(x,y) := \left(\frac{\partial P(x,y)}{\partial x_0}, \frac{\partial P(x,y)}{\partial y_0}, \ldots, \frac{\partial P(x,y)}{\partial x_n}, \frac{\partial P(x,y)}{\partial y_n} \right)$ and let us denote by $\deg_0 \nabla H$ the topological degree of the map $\epsilon \frac{\nabla H}{\|\nabla H\|} : S_\epsilon^{2n+1} \to S_\epsilon^{2n+1}$, for all $\epsilon > 0$ small enough. Now, it is easy to see that

$$\deg_0 \nabla P = (-1)^{n+1} \deg_0 \nabla H$$

and so Milnor's formula (2.6) follows from Theorem 2.11, item (i).

It seems natural to search for a similar result for non-isolated singularities. This was done in [13] and we will explain below the main strategy used to get an analogous formula.

Let us consider $F = (f_1, \ldots, f_p) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $1 \leq p \leq n-1$, an analytic map germ. Consider $l \in \{1, \ldots, p\}$ and $I = \{i_1, \ldots, i_l\}$ an l-tuple of pairwise distinct elements of $\{1, \ldots, p\}$, and let us denote by f_I the mapping $(f_{i_1}, \ldots, f_{i_l}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^l, 0)$. Suppose that F satisfies Milnor's condition (a) at the origin. Then, we have

$$\Sigma_{f_I} \subset \Sigma_F \subset F^{-1}(0) \subset f_I^{-1}(0),$$

and so by definition the map f_I also satisfies Milnor's condition (a) at the origin.

It is clear that $\Sigma_{f_I,\rho} \subset \Sigma_{F,\rho}$. In [13] it was proved the following key result:

Lemma 2.13 ([13]). *Assume that F satisfies Milnor's conditions (a) and (b) at the origin. Then, for all $l \in \{1,\dots,p\}$ and $I = \{i_1,\dots,i_l\} \subset \{1,\dots,p\}$, the map $f_I : (\mathbb{R}^n,0) \to (\mathbb{R}^l,0)$ satisfies Milnor's conditions (a) and (b).*

Proof. See [13, Lemma 4.1, page 7]. □

Corollary 2.14. *There exists $\epsilon_0 > 0$ such that, for all $l \in \{1,\dots,p\}$ and $I = \{i_1,\dots,i_l\} \subset \{1,\dots,p\}$, the maps $f_I : (\mathbb{R}^n,0) \to (\mathbb{R}^l,0)$ have ϵ_0 as a Milnor radius. Therefore, for all $2 \le l \le p$, we have the Milnor fibrations (see Theorem 2.3 and Corollary 2.4) for the map $f_I : (\mathbb{R}^n,0) \to (\mathbb{R}^l,0)$.*

In [13] several formulae were proved connecting the Euler-Poincaré characteristics of the links of the sets $f_I^{-1}(0)$ and of the Milnor fiber $\mathcal{M}_F^T$ for an analytic map germ F under Milnor's conditions (a) and (b). We remind below some of them.

As above, let us choose $l \in \{1,\dots,p\}$ and an l-tuple $I = \{i_1,\dots,i_l\}$ of pairwise distinct elements of $\{1,\dots,p\}$. We write $J = \{i_1,\dots,i_{l-1}\}$. We also denote by $\mathrm{Lk}^0(V_I)$ (resp. $\mathrm{Lk}^0(V_J)$) the local link of the zero set of f_I (resp. f_J). If $l = 1$ then $J = \emptyset$ and we put $f_J = 0$.

Proposition 2.15. *We have:*

$$\chi(\mathrm{Lk}^0(V_J)) - \chi(\mathrm{Lk}^0(V_I)) = (-1)^{n-l} 2\chi(\mathcal{M}_F^T).$$

Proof. See [13, Proposition 7.1, page 10]. □

Corollary 2.16. *Let $j \in \{1,\dots,p\}$. If n is even, then we have $\chi(\mathrm{Lk}^0(V_{\{j\}})) = 2\chi(\mathcal{M}_F^T)$ and if n is odd, then we have $\chi(\mathrm{Lk}^0(V_{\{j\}})) = 2 - 2\chi(\mathcal{M}_F^T)$.*

Proof. We apply the previous proposition to the case $l = 1$. In this case, if n is even then $\chi(\mathrm{Lk}^0(V_J)) = 0$ and if n is odd then $\chi(\mathrm{Lk}^0(V_J)) = 2$. □

Example 2.17. Consider $F : (\mathbb{C}^3,0) \to (\mathbb{C},0)$, $F(x,y,z) = x^2z + y^2$. It follows from Example 2.5 that Milnor's conditions (a) and (b) are clearly satisfied. In this case we can apply Sakamoto's formula [29] to get that the Milnor fiber $\mathcal{M}_F^T$ has the homotopy type of the 2-dimensional sphere S^2, and so $\chi(\mathcal{M}_F^T) = 2$. Let $P = \Re(F) : (\mathbb{R}^6,0) \to (\mathbb{R},0)$ be the function given by the real part of F. Observe that, since $\dim(\Sigma_P) > 0$, the link may not necessarily be a manifold, therefore we cannot find easily the Euler characteristic of the link. Applying our Corollary 2.16, we have that $\chi(\mathcal{L}_P) = 2\chi(\mathcal{M}_F^T) = 4$, where $\mathcal{L}_P := P^{-1}(0) \cap S_\epsilon^5$ is the link of the real function P.

Let us explain how to apply this result in order to get a formula expressing $\chi(\mathcal{M}_F^T)$ in terms of a topological degree.

Given analytic functions $h_1, \ldots, h_s : U \subseteq \mathbb{R}^n \to \mathbb{R}$, defined on an open neighborhood of the origin U, with $h_1(0) = \cdots = h_s(0) = 0$, let $h(x) = h_1(x)^2 + \cdots + h_s(x)^2$. Of course, the following equality of analytic sets holds:

$$\{x \in U : \ h_1(x) = \cdots = h_s(x) = 0\} = \{x \in U : \ h(x) = 0\}.$$

Szafraniec in [30] considered the function $g(x) = h(x) - c\rho(x)^k$, where $c > 0$ and k is an integer, and showed that for k large enough the function g has an isolated singular point at the origin. Moreover, he proved that for all small radius $\epsilon > 0$, the following Poincaré-Hopf type formula holds true:

$$\chi(\{x \in S_\epsilon^{n-1} : \ h(x) = 0\}) = 1 - \deg_0 \nabla g. \tag{2.8}$$

From Szafraniec's equation above and Corollary 2.16, we can state a Poincaré-Hopf type formula for local non-isolated singularities as follows:

Proposition 2.18. *Let $F = (f_1, \ldots, f_p) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $1 \leq p \leq n-1$, be an analytic map germ satisfying Milnor's conditions (a) and (b). Then there exist $c > 0$ and an integer k such that for $g(x) = f_1^2(x) - c\rho(x)^k$ we have,*

(i) if n is even, then $\chi(\mathcal{M}_F^T) = \frac{1}{2}(1 - \deg_0 \nabla g)$,

(ii) if n is odd, then $\chi(\mathcal{M}_F^T) = \frac{1}{2}(1 + \deg_0 \nabla g)$.

Proof. It follows from Corollary 2.16 and Szafraniec's results. □

§ 3. Fibration structure and degree formulae - Global setting

§ 3.1. (Global Milnor) Fibration structure

It is well known that, for a polynomial holomorphic function $f : \mathbb{C}^{n+1} \to \mathbb{C}$, there exists a finite set of complex values called the bifurcation set, denoted by $B_f \subset \mathbb{C}$, such that the restriction map $f_| : \mathbb{C}^{n+1} \setminus f^{-1}(B_f) \to \mathbb{C} \setminus B_f$ is a smooth locally trivial fibration (see for example Theorem 1 in [16]). If one takes a closed disc of radius R centered at the origin in the target space, i.e., $D_R \subset \mathbb{C}$ with $R > 0$ big enough in such a way that B_f is included inside its interior, this fibration induces the so-called *global monodromy fibration of* f,

$$f_| : \mathbb{C}^{n+1} \cap f^{-1}(S_R^1) \to S_R^1 \tag{3.1}$$

which is also a smooth locally trivial fibration, where S^1_R is the circle centered at the origin with radius R in $\mathbb{C}$.

In [26] Némethi and Zaharia considered a kind of regularity condition called *semi-tame condition* which controls the asymptotic behavior of the fibers at infinity. Following Milnor's method for the local case, they proved that for all radii R big enough, denoted by $R \gg 1$, the canonical (Milnor) projection

$$\frac{f}{|f|} : S^{2n+1}_R \setminus \{f = 0\} \to S^1 \tag{3.2}$$

is a smooth locally trivial fibration. But, as the authors explained in the introduction in [26] (see Section 4 in [26], for further results), one can check in Broughton's example $f(x, y) = x + x^2y$ that the fiber of the Milnor projection (3.2) is the thrice punctured 2-sphere and the generic fiber of (3.1) is the twice punctured 2-sphere. So, these two fibrations cannot be fiber-equivalent.

In the real case, this fibration structure has been also approched in [3] and more recently in a more general way by the authors of the present paper in [14]. Below we give the main ideas and results of [3].

Let $F = (f_1, \dots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$, $2 \leq p \leq n-1$, be a polynomial mapping, $V = F^{-1}(0)$ and for $R \gg 1$, denote by $\mathrm{Lk}^\infty(V) := V \cap S^{n-1}_R$ the link of V at infinity. Let us denote by $\rho : \mathbb{R}^n \to \mathbb{R}$, $\rho(x) = x_1^2 + \dots + x_n^2$, and so $S^{n-1}_R = \rho^{-1}(R^2)$.

Consider the (global Milnor's) projection $\frac{F}{\|F\|} : S^{n-1}_R \setminus \mathrm{Lk}^\infty(V) \to S^{p-1}$ and, as in the local case, also denote by:

1. Σ_F the set of critical points of F.
2. $M(\frac{F}{\|F\|})$ the set of critical points of the pair $(\frac{F}{\|F\|}, \rho)$ inside $\mathbb{R}^n \setminus V$.
3. $M(F)$ the critical locus of the pair (F, ρ).

With such a definition, the authors in [3] proved a fibration structure on spheres of radii big enough, as follows.

Following the definition stated in [3], we will consider the following conditions:

Condition (A): $\overline{M(F) \setminus V} \cap V$ is bounded in $\mathbb{R}^n$.

Condition (B): $M(\frac{F}{\|F\|})$ is bounded in $\mathbb{R}^n$.

Theorem 3.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^p$ be a real polynomial map such that $codim_{\mathbb{R}} V = p$ at infinity*[1] *and assume that Condition (A) holds. Then the following two are equivalent:*

[1] For the zero locus $V = F^{-1}(0) \subset \mathbb{R}^n$, we say that "$V$ has codimension p at infinity" if for any radius $R \gg 1$, $\mathrm{codim}_{\mathbb{R}}(V \setminus B_R) = p$, in the sense that every irreducible component of V has this property.

(i) $(\mathrm{Lk}^{\infty}(V), \frac{F}{\|F\|})$ *is a higher open book structure with singular binding on* S_R^{n-1} *for* $R \gg 1$.

(ii) Condition (B) holds.

Proof. We follow closely the proof given in [3]. We should say that the same idea of this proof was used in [14], but to prove a fibration theorem in a more general setting.

Firstly, see that Condition (B) holds if and only if for all $R \gg 1$ the projection mapping

$$\frac{F}{\|F\|} : S_R^{n-1} \setminus \mathrm{Lk}^{\infty}(V) \to S^{p-1}$$

is a submersion.

Condition (A) means that: there exists $\delta > 0$ small enough and a closed disc B_δ^p centered at the origin such that

$$F_| : F^{-1}(B_\delta^p - \{0\}) \cap S_R^{n-1} \to B_\delta^p - \{0\}$$

is a surjective proper smooth submersion.

The implication (i) $\Rightarrow$ (ii) is trivial, because if we suppose that $\frac{F}{\|F\|} : S_R^{n-1} \setminus \mathrm{Lk}^{\infty}(V) \to S^{p-1}$ is a locally trivial fibration, then it is a submersion and so Condition (B) follows.

Let us prove the converse. We can assume that $\mathrm{Lk}^{\infty}(V)$ is not empty: in fact, in the case it is empty, Condition (A) is always satisfied and by Condition (B), we have that $\frac{F}{\|F\|} : S_R^{n-1} \to S^{p-1}$ is a proper submersion and so onto, since S^{p-1} is connected. Hence (i) follows by Ehresmann's theorem.

In the case $\mathrm{Lk}^{\infty}(V)$ is not empty, the projection $\frac{F}{\|F\|} : S_R^{n-1} \setminus \mathrm{Lk}^{\infty}(V) \to S^{p-1}$ is not proper and we cannot use Ehresmann's theorem directly. Nevertheless, the mapping $F_| : F^{-1}(B_\delta^p - \{0\}) \cap S_R^{n-1} \to B_\delta^p - \{0\}$ is a surjective proper submersion. So, by Ehresmann's theorem, the mapping

$$F_| : F^{-1}(B_\delta^p - \{0\}) \cap S_R^{n-1} \to B_\delta^p - \{0\} \tag{3.3}$$

is a locally trivial fibration. Now, we can compose it with the radial projection $\pi_1 : B_\delta^p - \{0\} \to S^{p-1}$, $\pi_1(y) = \frac{y}{\|y\|}$, to get that

$$\frac{F}{\|F\|} : F^{-1}(B_\delta^p - \{0\}) \cap S_R^{n-1} \to S^{p-1} \tag{3.4}$$

is a locally trivial fibration, as well.

Fibration (3.3) yields that the map

$$\frac{F}{\|F\|} : F^{-1}(S_\delta^{p-1}) \cap S_R^{n-1} \to S^{p-1} \tag{3.5}$$

is also a locally trivial fibration and is surjective.

This implies that

$$\frac{F}{\|F\|} : S_R^{n-1} \setminus F^{-1}(\mathring{B}_\delta^p) \to S^{p-1} \tag{3.6}$$

is surjective and proper, where $\mathring{B}_\delta^p$ denotes the open disk for some $0 < \delta \ll \frac{1}{R}$. So, by using Condition (B), we see that the mapping $\frac{F}{\|F\|}$ is a smooth submersion on the compact manifold $S_R^{n-1} \setminus F^{-1}(\mathring{B}_\delta^p)$ with boundary. Therefore, we have a locally trivial fibration by Ehresmann's theorem.

Now we can glue fibrations (3.4) and (3.6) along the common boundary $F^{-1}(S_\delta^{p-1}) \cap S_R^{n-1}$, using fibration (3.5), to get the item (i). $\square$

Remark. As mentioned above, inspired by (the local) Milnor's conditions and the global conditions introduced in [3], the authors of the present paper in [14] introduced two conditions that produce open book structures on C^2 semi-algebraic manifolds.

§ 3.2. A degree formula in the complex case

In [8, 17, 25, 26] the authors considered a polynomial holomorphic function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ under the so-called tame, M-tame and semi-tame conditions. We do not intend to give more details in this direction, but we remind briefly the definitions of these conditions and give some topological results concerning the (global) Milnor fibers.

If one fixes coordinates $z = (z_0, \ldots, z_n)$, then one can consider the following quotient algebra

$$\mathcal{Q}_f = \frac{\mathbb{C}[z_0, \ldots, z_n]}{(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})},$$

where $\mathbb{C}[z_0, \ldots, z_n]$ denotes the ring of polynomial holomorphic functions and $(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})$ denotes the ideal spanned by the partial derivatives of f.

It is well known that f has only finitely many isolated singular points if and only if the complex dimension of the quotient algebra $\dim_{\mathbb{C}} \mathcal{Q}_f$ is finite (see for example [15]). This dimension is called the total Milnor number and it is denoted as in the local case by μ_f. As in the local case, the number μ_f is equal to the topological degree, denoted by $\deg_\infty \nabla f$, of the normalized gradient vector field

$$R\frac{\nabla f}{\|\nabla f\|} : S_R^{2n+1} \to S_R^{2n+1}, \tag{3.7}$$

where $R > 0$ is sufficiently large.

Denote by $\overline{\nabla}(f) = \left(\overline{\frac{\partial f}{\partial z_0}}, \ldots, \overline{\frac{\partial f}{\partial z_n}}\right)$ and the so-called (complex) Milnor set by $M(f) = \{z \in \mathbb{C}^{n+1}; \overline{\nabla}(f) = \lambda z \text{ for some } \lambda \in \mathbb{C}\}$.

Under the notations above, we say that a sequence $\{z_k\}$ satisfies the following condition $(*)$ if:

$$\{z_k\} \subset M(f),\ \|z_k\| \to +\infty \text{ and } \overline{\nabla}(f)(z_k) \to 0.$$

Definition 3.2. We say that a polynomial function f is:

(i) *M-tame* if for every sequence $\{z_k\}$ satisfying condition $(*)$, $|f(z_k)| \to +\infty$.

(ii) *semi-tame* if $c \in \mathbb{C}$ is such that there exists a sequence $\{z_k\}$ satisfying condition $(*)$ and such that $|f(z_k)| \to c$, then $c = 0$.

Of course, if f is M-tame, then it is semi-tame. But, the converse is not true, see for instance [26].

Theorem 3.3 ([25, 26] see also [17])**.**
Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be an M-tame holomorphic polynomial function. Then, the following holds true:

(i) the bifurcation set B_f is equal to the image of the singular locus $f(\Sigma_f)$,

(ii) for each value $w \in \mathbb{C} \setminus B_f$, the (typical) fiber $f^{-1}(w)$ has the homotopy type of a bouquet of μ_f n-dimensional spheres,

(iii) the global monodromy fibration of f (3.1) and the fibration (3.2) have diffeomorphic fibers.

So, the Broughton polynomial $f(z_0, z_1) = z_0(z_0 z_1 - 1)$ is not M-tame because $\Sigma_f = \emptyset$ but $B_f = \{0\}$.

It follows from item (ii) above that for such an M-tame polynomial we still have the following Poincaré-Hopf type formula:

$$\chi(F_\theta) = 1 + (-1)^n \mu_f,$$

where F_θ is the fiber of fibration (3.2).

§ 3.3. Topology at infinity of closed semi-algebraic sets and applications to global real Milnor fibers

Inspired by the formulae proved in local case and in the global complex case, we present below a formula connecting the Euler characteristic of the real Milnor fiber in a big sphere and the topological degree on a big sphere of a normalized gradient vector field. The strategy is the same as in the local case. First we need to establish global versions of Szafraniec's theorem [30]. Note that the results below are new.

3.3.1. On the link at infinity of a closed semi-algebraic set We establish several formulae relating the Euler characteristic of the link at infinity of a closed semi-algebraic set to a topological degree. If $X \subset \mathbb{R}^n$ is a semi-algebraic set, then we denote by $\mathrm{Lk}^\infty(X)$ its link at infinity.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2-semi-algebraic function. In the following, we will use results proved in [10] in the polynomial setting. But it is not difficult to see that these results hold in the C^2-semi-algebraic setting because the properties of polynomials and algebraic sets involved in [10] (finiteness of the number of critical values of a polynomial function, retract by deformation of a non-compact algebraic set to a compact semi-algebraic set...) are also satisfied in the C^2-semi-algebraic case. Furthermore, tools of differential topology (Morse theory, Ehresmann's theorem) are needed in [10], that is why we work in the C^2 category.

We assume that 0 is a critical value of f. We denote by $\Sigma_f = (\nabla f)^{-1}(0)$ the set of critical points of f. We remark that $\Sigma_f \cap f^{-1}(0)$ may not be compact.

Let $\omega(x) = 1 + \frac{1}{2}(x_1^2 + \cdots + x_n^2)$. Note that $\nabla\omega(x) = x$ and $\omega(x) \geq 1$. Let

$$\Gamma_{f,\omega} = \{x \in \mathbb{R}^n \mid \mathrm{rank}[\nabla f(x), \nabla\omega(x)] < 2\}.$$

We have $\Sigma_f \subset \Gamma_{f,\omega}$.

Lemma 3.4. *There is $k \in \mathbb{N}$ such that for all $x \in \Gamma_{f,\omega} \setminus f^{-1}(0)$,*

$$|f(x)| > \frac{1}{\omega(x)^k},$$

for $|x| \gg 1$.

Proof. Note that 1 is the greatest critical value of ω. We set $\tilde{S}_r = \omega^{-1}(r)$. Let $\beta :]1, +\infty[\to \mathbb{R}$ be defined by

$$\beta(r) = \inf\left\{|f(x)| \mid x \in \tilde{S}_r \cap (\Gamma_{f,\omega} \setminus f^{-1}(0))\right\}.$$

The function β is semi-algebraic. Furthermore $\beta > 0$ because for $r > 1$, $f_{|\tilde{S}_r}$ has a finite number of critical values. Thus the function $\frac{1}{\beta}$ is also semi-algebraic. Hence there exist $r_1 \geq 1$ and $k_0 \in \mathbb{N}$ such that $\frac{1}{\beta} < r^k$, for $r \geq r_1$ and $k \geq k_0$. This implies that $\beta(r) > \frac{1}{r^k}$ for $r \geq r_1$ and $k \geq k_0$. We can conclude that for $r \geq r_1$ and $k \geq k_0$,

$$|f(x)| > \frac{1}{\omega(x)^k},$$

for $x \in \tilde{S}_r \cap (\Gamma_{f,\omega} \setminus f^{-1}(0))$. □

Let $g_-(x) = f(x) - \frac{1}{\omega(x)^k}$. Note that $\Gamma_{f,\omega} = \Gamma_{g_-,\omega}$.

Lemma 3.5. *For $R \gg 1$, $\chi(\{g_- \leq 0\} \cap \tilde{S}_R) = \chi(\{f \leq 0\} \cap \tilde{S}_R)$.*

Proof. Let $R \gg 1$ be such that for all $x \in (\Gamma_{f,\omega} \setminus f^{-1}(0)) \cap \{\omega(x) \geq R\}$, $|f(x)| > \frac{1}{\omega(x)^k}$. Set $N_f^{\leq} = \{x \in \tilde{S}_R \mid f(x) \leq 0\}$ and $N_{g_-}^{\leq} = \{x \in \tilde{S}_R \mid g_-(x) \leq 0\}$. For $x \in \tilde{S}_R$, we have

$$g_-(x) \leq 0 \Leftrightarrow f(x) - \frac{1}{R^k} \leq 0 \Leftrightarrow f(x) \leq \frac{1}{R^k},$$

and so $N_f^{\leq} \subset N_{g_-}^{\leq}$. Furthermore if $0 < f(x) \leq \frac{1}{R^k}$ then $x \notin \Gamma_{f,\omega} \setminus f^{-1}(0)$ and therefore $\{f(x) \leq \frac{1}{R^k}\} \cap \tilde{S}_R$ retracts by deformation to $\{f(x) \leq 0\} \cap \tilde{S}_R$. We get the result. □

Corollary 3.6. *We have $\chi(\mathrm{Lk}^\infty(\{g_- \leq 0\})) = \chi(\mathrm{Lk}^\infty(\{f \leq 0\}))$.*

Let $g_+(x) = f(x) + \frac{1}{\omega(x)^k}$. Note that $\Gamma_{f,\omega} = \Gamma_{g_+,\omega}$.

Lemma 3.7. *For $R \gg 1$, $\chi(\{g_+ \geq 0\} \cap \tilde{S}_R) = \chi(\{f \geq 0\} \cap \tilde{S}_R)$.*

Proof. Let $R \gg 1$ be such that for all $x \in (\Gamma_{f,\omega} \setminus f^{-1}(0)) \cap \{\omega(x) \geq R\}$, $|f(x)| > \frac{1}{\omega(x)^k}$. Set $N_f^{\geq} = \{x \in \tilde{S}_R \mid f(x) \geq 0\}$ and $N_{g_+}^{\geq} = \{x \in \tilde{S}_R \mid g_+(x) \geq 0\}$. For $x \in \tilde{S}_R$, we have

$$g_+(x) \geq 0 \Leftrightarrow f(x) + \frac{1}{R^k} \geq 0 \Leftrightarrow f(x) \geq -\frac{1}{R^k},$$

and so $N_f^{\geq} \subset N_{g_+}^{\geq}$. Furthermore if $0 > f(x) \geq -\frac{1}{R^k}$ then $x \notin \Gamma_{f,\omega} \setminus f^{-1}(0)$ and therefore $\{f(x) \geq -\frac{1}{R^k}\} \cap \tilde{S}_R$ retracts by deformation to $\{f(x) \geq 0\} \cap \tilde{S}_R$. We get the result. □

Corollary 3.8. *We have $\chi(\mathrm{Lk}^\infty(\{g_+ \geq 0\})) = \chi(\mathrm{Lk}^\infty(\{f \geq 0\}))$.*

Lemma 3.9. *The sets $(\nabla g_-)^{-1}(0) \cap \{g_- = 0\}$ and $(\nabla g_+)^{-1}(0) \cap \{g_+ = 0\}$ are compact.*

Proof. If $x \in (\nabla g_-)^{-1}(0) \cap \{g_- = 0\}$ then $x \in \Gamma_{g_-,\omega} = \Gamma_{f,\omega}$ and $f(x) = \frac{1}{\omega(x)^k} \neq 0$. So $x \in \Gamma_{f,\omega} \setminus f^{-1}(0)$. But there exists $R \geq r_1$ such that $|f(x)| > \frac{1}{\omega(x)^k}$ if $\rho(x) \geq R$ and $x \in \Gamma_{f,\omega} \setminus f^{-1}(0)$. We conclude that

$$\nabla(g_-)^{-1}(0) \cap \{g_- = 0\} \subset \tilde{B}_R,$$

where $\tilde{B}_R = \{\omega \leq R\}$. □

We make the assumption that $f(0) > 1$.

Lemma 3.10. *We have $g_-(0) > 0$ and $g_+(0) > 0$.*

Proof. We have $g_-(0) = f(0) - 1$ and $g_+(0) = f(0) + 1$. □

Let H_- and $H_+ : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be defined by

$$H_-(x, \lambda) = (\lambda x + \nabla g_-, g_-) \text{ and } H_+(x, \lambda) = (\lambda x + \nabla g_+, g_+).$$

Lemma 3.11. *The sets $H_-^{-1}(0)$ and $H_+^{-1}(0)$ are compact.*

Proof. See [10], Lemma 5.4 and Remark 5.11. □

Hence there exists $R \gg 1$ such that $H_-^{-1}(0) \subsetneq B_R^{n+1}$. We will denote by $\deg_\infty H_-$ the topological degree of the mapping $\frac{H_-}{\|H_-\|} : S_R^n \to S^n$.

Let δ_- be a small regular value of g_-. Let $R_- \gg 1$ be such that $\{g_- = 0\} \cap \tilde{B}_{R_-}$ is a retract by deformation of $\{g_- = 0\}$, $\{g_- \leq 0\} \cap \tilde{B}_{R_-}$ is a retract by deformation of $\{g_- \leq 0\}$ and $\{g_- \geq 0\} \cap \tilde{B}_{R_-}$ is a retract by deformation of $\{g_- \geq 0\}$.

Proposition 3.12. *If n is even, we have*

$$\chi(\{g_- = \delta_-\} \cap \tilde{B}_{R_-}) = \deg_\infty H_-.$$

If n is odd, we have

$$\chi(\{g_- \geq \delta_-\} \cap \tilde{B}_{R_-}) - \chi(\{g_- \leq \delta_-\} \cap \tilde{B}_{R_-}) = 1 - \deg_\infty H_-.$$

Proof. Apply equalities **(A)** and **(D)** in [10], page 332, and use the fact that $g_-(0) > 0$. □

If n is even, we have

$$\chi(\{g_- = \delta_-\} \cap \tilde{B}_{R_-}) = \frac{1}{2}\chi(\{g_- = \delta_-\} \cap \tilde{S}_{R_-}) = \chi(\{g_- \leq \delta_-\} \cap \tilde{S}_{R_-}) =$$

$$\chi(\{g_- \leq 0\} \cap \tilde{S}_{R_-}) = \chi(\mathrm{Lk}^\infty(\{g_- \leq 0\})).$$

If n is odd, we have

$$\chi(\{g_- \geq \delta_-\} \cap \tilde{B}_{R_-}) - \chi(\{g_- \leq \delta_-\} \cap \tilde{B}_{R_-}) =$$

$$\frac{1}{2}\left[\chi(\{g_- \geq \delta_-\} \cap \tilde{S}_{R_-}) - \chi(\{g_- \leq \delta_-\} \cap \tilde{S}_{R_-})\right].$$

Since,

$$\chi(\tilde{S}_{R_-}) = 2 = \chi(\{g_- \geq \delta_-\} \cap \tilde{S}_{R_-}) + \chi(\{g_- \leq \delta_-\} \cap \tilde{S}_{R_-}),$$

we find that $\chi(\{g_- \geq \delta_-\} \cap \tilde{S}_{R_-}) = 2 - \chi(\{g_- \leq \delta_-\} \cap \tilde{S}_{R_-})$ and that

$$\chi(\{g_- \geq \delta_-\} \cap \tilde{B}_{R_-}) - \chi(\{g_- \leq \delta_-\} \cap \tilde{B}_{R_-}) = 1 - \chi(\{g_- \leq \delta_-\} \cap \tilde{S}_{R_-}) =$$

$$1 - \chi(\{g_- \leq 0\} \cap \tilde{S}_{R_-}) = 1 - \chi(\mathrm{Lk}^\infty(\{g_- \leq 0\})).$$

Corollary 3.13. *We have*

$$\chi(\mathrm{Lk}^\infty(\{g_- \leq 0\})) = \chi(\mathrm{Lk}^\infty(\{f \leq 0\})) = \deg_\infty H_-.$$

Let δ_+ be a small regular value of g_+. Let $R_+ \gg 1$ be such that $\{g_+ = 0\} \cap \tilde{B}_{R_+}$ is a retract by deformation of $\{g_+ = 0\}$, $\{g_+ \leq 0\} \cap \tilde{B}_{R_+}$ is a retract by deformation of $\{g_+ \leq 0\}$ and $\{g_+ \geq 0\} \cap \tilde{B}_{R_+}$ is a retract by deformation of $\{g_+ \geq 0\}$.

Proposition 3.14. *If n is even, we have*

$$\chi(\{g_+ = \delta_+\} \cap \tilde{B}_{R_+}) = \deg_\infty H_+.$$

If n is odd, we have

$$\chi(\{g_+ \geq \delta_+\} \cap \tilde{B}_{R_+}) - \chi(\{g_+ \leq \delta_+\} \cap \tilde{B}_{R_+}) = 1 - \deg_\infty H_+.$$

Proof. Apply equalities **(A)** and **(D)** in [10], page 332, and use the fact that $g_+(0) > 0$. □

Corollary 3.15. *If n is even, we have*

$$\chi(\mathrm{Lk}^\infty(\{f \geq 0\})) = \deg_\infty H_+.$$

If n is odd, we have

$$\chi(\mathrm{Lk}^\infty(\{f \geq 0\})) = 2 - \deg_\infty H_+.$$

Proof. If n is even, we apply the same proof as for $\mathrm{Lk}^\infty(\{f \leq 0\})$.

If n is odd, we remark that

$$\chi(\{g_+ \geq \delta_+\} \cap \tilde{B}_{R_+}) - \chi(\{g_+ \leq \delta_+\} \cap \tilde{B}_{R_+}) =$$
$$\chi(\{g_+ \geq \delta_+\} \cap \tilde{S}_{R_+}) - 1 = \chi(\mathrm{Lk}^\infty(\{f \geq 0\})) - 1 = 1 - \deg_\infty H_+.$$

□

Corollary 3.16. *If n is even, we have*

$$\chi(\mathrm{Lk}^\infty(\{f = 0\})) = \deg_\infty H_- + \deg_\infty H_+.$$

If n is odd, we have

$$\chi(\mathrm{Lk}^\infty(\{f = 0\})) = \deg_\infty H_- - \deg_\infty H_+.$$

Proof. Use the equality

$$\chi(S^{n-1}) = \chi(\mathrm{Lk}^\infty(\{f \geq 0\})) + \chi(\mathrm{Lk}^\infty(\{f \leq 0\})) - \chi(\mathrm{Lk}^\infty(\{f = 0\})),$$

and the previous corollaries. □

Application: Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set. By Theorem C.11 in [31], there exists a C^2-semi-algebraic function $f : \mathbb{R}^n \to \mathbb{R}$ such that $X = f^{-1}(0)$. Furthermore by Corollary C.12 in [31], we can assume that $f \geq 0$. By a change of origin if necessary, we can assume that $f(0) > 1$. In this case, $g_+ > 0$ and so $\deg_\infty H_+ = 0$.

Corollary 3.17. *We have* $\chi(\mathrm{Lk}^\infty(X)) = \deg_\infty H_-$.

Now we return to the general case. The functions g_- and g_+ are semi-algebraic but not polynomial even if f is a polynomial. Let G_- and G_+ be defined by

$$G_-(x) = \omega(x)^k g_-(x) = \omega(x)^k f(x) - 1,$$

and

$$G_+(x) = \omega(x)^k g_+(x) = \omega(x)^k f(x) + 1.$$

If f is a polynomial then so are G_- and G_+. Furthermore, we have $\{G_- = 0\} = \{g_- = 0\}$, $\{G_- \leq 0\} = \{g_- \leq 0\}$, $\{G_- \geq 0\} = \{g_- \geq 0\}$ and $\{G_+ = 0\} = \{g_+ = 0\}$, $\{G_+ \leq 0\} = \{g_+ \leq 0\}$, $\{G_+ \geq 0\} = \{g_+ \geq 0\}$.

Lemma 3.18. *The sets* $(\nabla G_-)^{-1}(0) \cap \{G_- = 0\}$ *and* $(\nabla G_+)^{-1}(0) \cap \{G_+ = 0\}$ *are compact.*

Proof. We have

$$\nabla G_-(x) = k\omega(x)^{k-1} g_-(x) \nabla\omega(x) + \omega(x)^k \nabla g_-(x).$$

If $G_-(x) = 0$ then $g_-(x) = 0$ and so $\nabla G_-(x) = \omega(x)^k \nabla g_-(x)$. We see that $(\nabla G_-)^{-1}(0) \cap \{G_- = 0\} = (\nabla g_-)^{-1}(0) \cap \{g_- = 0\}$, which is compact. □

Let L_- and $L_+ : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be defined by

$$L_-(x,\lambda) = (\lambda x + \nabla G_-, G_-) \text{ and } L_+(x,\lambda) = (\lambda x + \nabla G_+, G_+).$$

Lemma 3.19. *The sets* $L_-^{-1}(0)$ *and* $L_+^{-1}(0)$ *are compact.*

Proof. See [10], Lemma 5.4 and Remark 5.11. □

Corollary 3.20. *We have*

$$\chi(\mathrm{Lk}^\infty(\{f \leq 0\})) = \deg_\infty L_-,$$

$$\chi(\mathrm{Lk}^\infty(\{f \geq 0\})) = \deg_\infty L_+, \text{if } n \text{ is even},$$

$$\chi(\mathrm{Lk}^\infty(\{f \geq 0\})) = 2 - \deg_\infty L_+, \text{if } n \text{ is odd},$$

$$\chi(\mathrm{Lk}^\infty(\{f = 0\})) = \deg_\infty L_- + \deg_\infty L_+, \text{if } n \text{ is even},$$

$$\chi(\mathrm{Lk}^\infty(\{f = 0\})) = \deg_\infty L_- - \deg_\infty L_+, \text{if } n \text{ is odd}.$$

Application: Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set. There exists a C^2-semi-algebraic function $f : \mathbb{R}^n \to \mathbb{R}$ such that $X = f^{-1}(0)$ and $f \geq 0$. By a change of origin if necessary, we can assume that $f(0) > 1$. In this case, $G_+ > 0$ and so $\deg_\infty L_+ = 0$.

Corollary 3.21. *We have* $\chi(\mathrm{Lk}^\infty(X)) = \deg_\infty L_-$.

Next we focus on the case when the C^2 semi-algebraic function is semi-tame.

Definition 3.22. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2-semi-algebraic function. We say that f is semi-tame if for any sequence of points $(x_n)_{n\in\mathbb{N}}$ in $\Gamma_{f,\omega}$ such that $|x_n| \to +\infty$ and $\lim_{n\to+\infty} |f(x_n)| = l \in \mathbb{R}^+$, we have $l = 0$.

For example, any weighted homogeneous polynomial is semi-tame. A proof of this fact is given in [26], page 329, in the complex case, but it is not difficult to see that the arguments work in the real case as well.

In the following, we assume that f is semi-tame.

Lemma 3.23. *The set* $(\nabla f)^{-1}(0) \setminus \{f = 0\}$ *is compact.*

Proof. Since f is C^2 and semi-algebraic, its set of critical values is finite. A proof of this fact is given in Theorem 2.5.11 in [6] for C^∞ functions. Using Assertion 4.8 in [31], we see that this proof also works for C^2 functions. Let $\alpha \neq 0$ be a critical value of f. Let $x \in f^{-1}(\alpha) \cap (\nabla f)^{-1}(0)$, x belongs to $\Gamma_{f,\omega}$. If $f^{-1}(\alpha) \cap \nabla f^{-1}(0)$ is not compact then there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ such that $|x_n| \to +\infty$, $f(x_n) = \alpha$ and $\nabla f(x_n) = 0$. But $x_n \in \Gamma_{f,\omega}$, this contradicts the semi-tameness of f. $\square$

Lemma 3.24. *The functions* g_- *and* g_+ *are semi-tame.*

Proof. We know that $\Gamma_{g_-,\omega} = \Gamma_{f,\omega}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in $\Gamma_{g_-,\omega}$ such that $|x_n| \to +\infty$ and $\lim_{n\to+\infty} |g_-(x_n)| = l$. Then $x_n \in \Gamma_{f,\omega}$, $|x_n| \to +\infty$ and $\lim_{n\to+\infty} |f(x_n)| = l$ since $g_-(x_n) = f(x_n) - \frac{1}{\omega(x_n)^k}$. But f is semi-tame so $l = 0$. $\square$

Corollary 3.25. *The sets* $(\nabla g_-)^{-1}(0)$ *and* $(\nabla g_+)^{-1}(0)$ *are compact.*

Proof. The function g_- is semi-tame so $(\nabla g_-)^{-1}(0) \setminus \{g = 0\}$ is compact. We know that $(\nabla g_-)^{-1}(0) \cap \{g = 0\}$ is compact. $\square$

In [12], the author introduced three sets $\Lambda_h^{\leq}$, $\Lambda_h^{=}$ and $\Lambda_h^{\geq}$ for a C^2 semi-algebraic function h. They are defined as follows: for $* \in \{\leq, =, \geq\}$,

$$\Lambda_h^* = \Big\{\alpha \in \mathbb{R} \mid \beta \mapsto \chi\big(\mathrm{Lk}^\infty(X \cap \{h * \beta\})\big) \text{ is not constant in a neighborhood of } \alpha\Big\}.$$

By [12], Lemma 3.12, we have $\Lambda_{g_-}^{\leq} \subset \{0\}$, $\Lambda_{g_-}^{\geq} \subset \{0\}$ and $\Lambda_{g_-}^{=} \subset \{0\}$ since g_- is semi-tame.

Proposition 3.26. *Let* $\alpha_- \in]-\infty, 0[$ *and* $\alpha_+ \in]0, +\infty[$. *We have*

$$\chi(\mathrm{Lk}^\infty(\{g_- \leq \alpha_-\})) + \chi(\mathrm{Lk}^\infty(\{g_- \leq \alpha_+\})) - \chi(\mathrm{Lk}^\infty(\{g_- \leq 0\})) =$$

$$1 - \deg_\infty \nabla g_-.$$

Proof. See [12], Theorem 3.16. □

Lemma 3.27. *Let* $\alpha \neq 0$. *We have*

$$\chi(\mathrm{Lk}^\infty(\{g_- \leq \alpha\})) = \chi(\mathrm{Lk}^\infty(\{f \leq \alpha\})).$$

Proof. We assume first that $\alpha < 0$. Furthermore, we can suppose that $\frac{1}{2}\alpha + 1 < 0$ because the function $\beta \mapsto \chi\big(\mathrm{Lk}^\infty(X \cap \{f \leq \beta\})\big)$ is constant on $]-\infty, 0[$ by the semi-tameness of f. The function g_- is semi-tame so if $(x_n)_{n\in\mathbb{N}}$ is a sequence of points in $\Gamma_{g_-,\omega}$ such that $|x_n| \to +\infty$ and $g_-(x_n) \leq \frac{1}{2}\alpha$ then $\lim_{n\to+\infty} g_-(x_n) = -\infty$. Since $\Gamma_{f,\omega} = \Gamma_{g_-,\omega}$ and f is semi-tame, we also have $\lim_{n\to+\infty} f(x_n) = -\infty$ because $f(x_n) \leq \frac{1}{2}\alpha + 1 < 0$. So there exits $R_0 \gg 1$ such that for all $R \geq R_0$, $f(x) \leq 2\alpha$ and $g_-(x) \leq 2\alpha$ for $x \in \tilde{S}_R \cap \Gamma_{g_-,\omega} \cap \{g_- \leq \frac{1}{2}\alpha\}$. So $\mathrm{Lk}^\infty(\{g_- \leq \alpha\})$ is homeomorphic to $\{g_- \leq \alpha\} \cap \tilde{S}_R$ and $\mathrm{Lk}^\infty(\{f \leq \alpha\})$ is homeomorphic to $\{f \leq \alpha\} \cap \tilde{S}_R$ for any $R \geq R_0$. But $\{g_- \leq \alpha\} \cap \tilde{S}_R = \{f \leq \alpha + \frac{1}{R^k}\}) \cap \tilde{S}_R$. Since $\tilde{S}_R \cap \Gamma_{f,\omega} \cap \{\alpha \leq f \leq \alpha + \frac{1}{R^k}\}$ is empty, the set $\{f \leq \alpha + \frac{1}{R^k}\} \cap \tilde{S}_R$ is homeomorphic to $\{f \leq \alpha\} \cap \tilde{S}_R$.

If $\alpha > 0$ the proof is the same, replacing $\{g_- \leq \frac{1}{2}\alpha\}$ with $\{g_- \geq \frac{1}{2}\alpha\}$ and taking R such that $\alpha + \frac{1}{R^k} < 2\alpha$. □

Corollary 3.28. *Let* $\alpha_- \in]-\infty, 0[$ *and* $\alpha_+ \in]0, +\infty[$. *We have*

$$\chi(\mathrm{Lk}^\infty(\{f \leq \alpha_-\})) + \chi(\mathrm{Lk}^\infty(\{f \leq \alpha_+\})) - \chi(\mathrm{Lk}^\infty(\{f \leq 0\})) =$$

$$1 - \deg_\infty \nabla g_-.$$

Similarly, we can state:

Corollary 3.29. *Let* $\alpha_- \in]-\infty, 0[$ *and* $\alpha_+ \in]0, +\infty[$. *We have*

$$\chi(\mathrm{Lk}^\infty(\{f \geq \alpha_-\})) + \chi(\mathrm{Lk}^\infty(\{f \geq \alpha_+\})) - \chi(\mathrm{Lk}^\infty(\{f \geq 0\})) =$$

$$1 - (-1)^n \deg_\infty \nabla g_+.$$

Corollary 3.30. *Let* $\alpha_- \in]-\infty, 0[$ *and* $\alpha_+ \in]0, +\infty[$. *We have*

$$\chi(\mathrm{Lk}^\infty(\{f = \alpha_-\})) + \chi(\mathrm{Lk}^\infty(\{f = \alpha_+\})) - \chi(\mathrm{Lk}^\infty(\{f = 0\})) =$$

$$2 - \chi(S^{n-1}) - (\deg_\infty \nabla g_- + (-1)^n \deg_\infty \nabla g_+).$$

Proof. Use the previous two corollaries and the Mayer-Vietoris sequence. □

Application: Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set. There exists a C^2-semi-algebraic function $f : \mathbb{R}^n \to \mathbb{R}$ such that $X = f^{-1}(0)$ and $f \geq 0$. By a change of origin if necessary, we can assume that $f(0) > 1$.

Lemma 3.31. *There exists a C^2-semi-algebraic function $\Phi : \mathbb{R}^n \to \mathbb{R}$ such that*

1. $X = \Phi^{-1}(0)$ *and* $\Phi \geq 0$,
2. *for any sequence of points $(x_n)_{n\in\mathbb{N}}$ in $\Gamma_{\Phi,\omega} \setminus X$ such that $|x_n| \to +\infty$, we have* $\lim_{n\to+\infty} |\Phi(x_n)| = +\infty$.

Moreover one can take $\Phi = \omega^K f$ with $K \gg 1$.

Proof. Same proof as Lemma 3.4 (see also [11], Proposition 7.1). □

Corollary 3.32. *The function Φ is semi-tame.*

Let us remark that

1. if $\alpha_- < 0$, then the sets $\{\Phi \leq \alpha_-\}$ and $\{\Phi = \alpha_-\}$ are empty,
2. if $\alpha_- < 0$, then we have $\{\Phi \geq \alpha_-\} = \{\Phi \geq 0\}$,
3. we have $\{\Phi \leq 0\} = \{\Phi = 0\}$,
4. if $\alpha_+ > 0$ is small enough, then $\mathrm{Lk}^\infty(\{\Phi \leq \alpha_+\})$ retracts by deformation to $\mathrm{Lk}^\infty(\{\Phi = 0\})$.

In this situation, the above corollaries applied to Φ become the following corollaries.

Corollary 3.33. *We have* $\deg_\infty \nabla g_- = 1$.

Corollary 3.34. *We have*

$$\chi(\mathrm{Lk}^\infty(\{\Phi \geq \alpha_+\})) = 1 - (-1)^n \deg_\infty \nabla g_+.$$

Corollary 3.35. *We have*

$$\chi(\mathrm{Lk}^\infty(\{\Phi = \alpha_+\})) - \chi(\mathrm{Lk}^\infty(X)) = 1 - \chi(S^{n-1}) - (-1)^n \deg_\infty \nabla g_+.$$

Here we can state the global version of Szafraniec's theorem [30].

Corollary 3.36. *We have*

$$\chi(\mathrm{Lk}^{\infty}(X)) = 1 - \deg_{\infty} \nabla g_{+}.$$

Proof. If n is odd, $\mathrm{Lk}^{\infty}(\{\Phi = \alpha_{+}\})$ is a compact odd-dimensional manifold and $\chi(S^{n-1}) = 2$.

If n is even, $\chi(S^{n-1}) = 0$. Moreover

$$\chi(\mathrm{Lk}^{\infty}(\{\Phi = \alpha_{+}\})) = 2\chi(\mathrm{Lk}^{\infty}(\{\Phi \geq \alpha_{+}\})) = 2 - 2\deg_{\infty} \nabla g_{+}.$$

□

3.3.2. Applications for the global Milnor fiber on the spheres

Let $F = (f_1, \cdots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$ be a C^2 semi-algebraic mapping such that Conditions (A) and (B) hold. Let $\mathcal{M}^S_{\frac{F}{\|F\|}}$ be the fiber of the locally trivial fibration $\frac{F}{\|F\|} : S_R^{n-1} \setminus \mathrm{Lk}^{\infty}(V) \to S^{p-1}$, where $V = F^{-1}(0)$ and $R \gg 1$. In [14] Corollary 4.3, the authors proved a global version of Corollary 2.16.

Proposition 3.37. *Let $j \in \{1, \ldots, p\}$. If n is even, then we have*

$$\chi(\mathrm{Lk}^{\infty}(f_j^{-1}(0)) = 2\chi(\mathcal{M}^S_{\frac{F}{\|F\|}}),$$

and if n is odd, then we have

$$\chi(\mathrm{Lk}^{\infty}(f_j^{-1}(0)) = 2 - 2\chi(\mathcal{M}^S_{\frac{F}{\|F\|}}).$$

Example 3.38. Let $F : \mathbb{C}^2 \to \mathbb{C}$, $F(x, y) = x(x + y)\bar{x}$. From [3], this example is a polar weighted homogenous mixed polynomial which verifies Conditions (A) and (B). Therefore the fibration $\frac{F}{\|F\|}$ exists. The fiber is homotopy equivalent to $\mathbb{R}^3 \setminus \mathbb{R}$ and the Euler characteristic of the fiber is 0. On the other hand, we can write F as a polynomial map $F : \mathbb{R}^4 \to \mathbb{R}^2$, $F(a, b, c, d) = ((a^2 + b^2)(a + c), (a^2 + b^2)(b + d))$. Hence the Euler characteristic of the link $\chi(\mathrm{Lk}^{\infty}((a^2 + b^2)(a + c) = 0)) = 0$, by using our previous proposition, we get that the Euler characteristic of the fiber is 0.

Example 3.39. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$, $F(x, y, z) = (x^2 + y, x + z)$. Then F does not have any singularity so by Lemma 3.5 in [3], Condition (A) is satisfied. Moreover by computation, the set $M(\frac{F}{\|F\|})$ is empty. Therefore F verifies Condition (B) and the fibration $\frac{F}{\|F\|}$ exists. The fiber is homeomorphic to an arc of a circle which has Euler characteristic equal to 1. On the other hand $\chi(\mathrm{Lk}^{\infty}(x + z = 0)) = 0$, by using our previous proposition, we get that the Euler characteristic of the fiber is 1.

Applying Proposition 3.37 and Corollary 3.36, we can state a Poincaré-Hopf formula for $\mathcal{M}^S_{\frac{F}{\|F\|}}$.

Proposition 3.40. *Let $F = (f_1, \ldots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$, $1 \leq p \leq n-1$, be a C^2 semi-algebraic mapping such that Conditions (A) and (B) hold. Assume for instance that $f_1(0) > 1$. There exist two integers $K, k > 0$ such that if $g_+(x) = \omega(x)^K f_1^2(x) + \frac{1}{\omega(x)^k}$ then we have,*

(i) if n is even, then $\chi(\mathcal{M}^{S}_{\frac{F}{\|F\|}}) = \frac{1}{2}(1 - \deg_\infty \nabla g_+)$,

(ii) if n is odd, then $\chi(\mathcal{M}^{S}_{\frac{F}{\|F\|}}) = \frac{1}{2}(1 + \deg_\infty \nabla g_+)$.

References

[1] Araújo dos Santos, R., Uniform (m)-condition and strong Milnor fibrations, *Singularities II*, 189–198, Contemp. Math., **475**, Amer. Math. Soc., Providence, RI, 2008.

[2] Araújo dos Santos, R., Chen, Y. and Tibăr, M., Singular open book structures from real mappings, *Cent. Eur. J. Math.* **11** (2013), no. 5, 817–828.

[3] Araújo dos Santos, R., Chen, Y. and Tibăr, M., Real polynomial maps and singular open books at infinity, to appear in *Scandinavica Math.*

[4] Araújo dos Santos, R., Dreibelbis, D. and Dutertre, N., Topology of the real Milnor fiber for isolated singularities, *Real and complex singularities*, 67–75, Contemp. Math., **569**, Amer. Math. Soc., Providence, RI, 2012.

[5] Araújo dos Santos, R. and Tibăr, M., Real map germs and higher open book structures, *Geom. Dedicata* **147** (2010), 177–185.

[6] Benedetti, R. and Risler, J. J., *Real algebraic and semi-algebraic sets*, Hermann, Paris, (1990).

[7] Bochnak, J., Coste, M. and Roy, M. F., *Real algebraic geometry*, Ergeb. Math. Grenzgeb. (3) **36**, Springer-Verlag, Berlin (1998).

[8] Broughton, S. A., Milnor numbers and the topology of polynomial hypersurfaces, *Invent. Math.* **92** (1988), no. 2, 217–241.

[9] Cisneros-Molina, J. L., Seade, J. and Snoussi, J., Milnor fibrations and d-regularity for real analytic singularities, *Internat. J. Math.* **21** (2010), no. 4, 419–434.

[10] Dutertre, N., On topological invariants associated with a polynomial with isolated critical points, *Glasg. Math. J.* **46**, no. 2 (2004), 323–334.

[11] Dutertre, N., Semi-algebraic neighborhoods of closed semi-algebraic sets, *Annales de l'Institut Fourier* **59**, no.1 (2009), 429–458.

[12] Dutertre, N., On the topology of semi-algebraic functions on closed semi-algebraic sets, *Manuscripta Mathematica* **139**, no. 3-4 (2012), 415–441.

[13] Dutertre, N. and Araújo dos Santos, R., Topology of real Milnor fibration for non-isolated singularities, to appear in *International Mathematics Research Notices* 2015, DOI: 10.1093/imrn/rnv286.

[14] Dutertre, N., Araújo dos Santos, R., Chen, Y. and Andrade, A., Open book structures on semi-algebraic manifolds, to appear in *Manuscripta Mathematica* 2015, DOI: 10.1007/s00229-015-0772-4.

[15] Gonzalez-Vega, L., Rouillier, F., and Roy, M. F., Symbolic recipes for polynomial system solving, In: *Some tapas of computer algebra*, A. Cohen et al. ed. Springer, 34–65 (1999).

[16] Hà, H.V. and Lê, D.T., Sur la topologie des polynômes complexes, *Acta Math. Vietnam.* **9** (1984), no. 1, 21–32 (1985).

[17] Hà, H.V. and Zaharia, A., Families of polynomials with total Milnor number constant, *Math. Ann.* **304** (1996), no. 3, 481–488.

[18] Hamm, H., On the Euler characteristic of real Milnor Fibres, *Proceeding of 12th International Workshop on Singularities - 2012* "Celebrating the 60th birthday of Prof. Shyuichi Izumiya", *Journal of Singularities*, **10** (2014), 147–156, DOI: 10.5427/jsing.2014.10i

[19] Hamm, H. and Lê, D.T., Un théorème de Zariski du type de Lefschetz, *Ann. Sci. École Norm. Sup. (4)* **6** (1973), no. 3, 317–355.

[20] Khimshiashvili, G. M., On the local degree of a smooth map, *Soobshch. Akad. Nauk Gruz. SSR* **85** (1977), no. 2, 309–312.

[21] Lê, D.T., Some remarks on the relative monodromy, in *Real and Complex Singularities*, Proc. Nordic Summer School, Oslo 1976, Sijhoff en Nordhoff, Alphen a.d. Rijn 1977, 397–403.

[22] Łojasiewicz, S., *Ensembles semi-analytiques*, Inst. Hautes Études Sci., Bures-sur-Yvette, 1965.

[23] Massey, D., Real analytic Milnor fibrations and a strong Łojasiewicz inequality, *Real and complex singularities*, London Math. Soc. Lecture Note Ser., **380**, Cambridge Univ. Press, Cambridge (2010), 268–292.

[24] Milnor, J., *Singular points of complex hypersurfaces,* Ann. of Math. Studies **61**, Princeton University Press (1968).

[25] Némethi, A. and Zaharia, A., On the bifurcation set of a polynomial function and Newton boundary, *Publ. Res. Inst. Math. Sci.* **26** (1990), no. 4, 681–689.

[26] Némethi, A. and Zaharia, A., Milnor fibration at infinity, *Indag. Math. (N.S.)* **3** (1992), no. 3, 323–335.

[27] Oka, M., Non-degenerate mixed functions, *Kodai Math. J.* **33** (2010), no. 1, 1–62.

[28] Ruas, M. A. S. and Araújo dos Santos, R., Real Milnor fibrations and (c)-regularity, *Manuscripta Mathematica* **117** (2005), no. 2, 207–218.

[29] Sakamoto, K., Milnor fiberings and their characteristic maps, *Manifolds Tokyo 1973* (Proc. Internat. Conf., Tokyo, 1973), pp. 145–150. Univ. Tokyo Press, Tokyo, 1975.

[30] Szafraniec, Z., On the Euler characteristic of analytic and algebraic sets, *Topology* **26** (1986), 411–414.

[31] van den Dries, L. and Miller, C., Geometric categories and o-minimal structures, *Duke Math. J.* **84** (1996), no. 2, 497–540.